Price discovery and sentiment☆

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1. Introduction

Easley and O’Hara’s (1992) seminal paper serves as a milestone in the market microstructure literature. In their paper, Easley and O’Hara pioneered the notion of a time domain of trades, i.e., time per se plays a role in the adjustment of security prices to the new information that trading reveals to the market. One of the most important contributions of their model is that it provides a characterization of the underlying stochastic process of security prices and thus gives rise to the important divergences that arise between transaction prices and this underlying stochastic process. The current paper revisits the market assumed by Easley and O’Hara, with an emphasis on the consequences of considering investor sentiment.

In their model, Easley and O’Hara (1992) assume two categories of traders; informed or uninformed. Informed traders possess an informational advantage from observing a private-information signal. When facing a positive signal, informed traders submit buy orders and when the signal is negative they submit sell orders. Uninformed traders don’t observe any information signal, and trade in the market simply for liquidity or price-sensitivity reasons. Unlike informed traders, uninformed traders submit both buy and sell orders independent of any relevant information. In Easley and O’Hara’s model, the interaction between asymmetrically-informed traders results in informed traders using their informational advantage to profit at the expense of uninformed traders. Market makers, who are also uninformed, charge the bid-ask spread against the risk of losses to informed traders.

Easley and O’Hara’s (1992) clear dividing line between informed and uninformed traders is very useful in studying the impact of asymmetric information on the formation of the bid-ask spread and price efficiency. However, this formulation can be improved to better reflect the complexity of real markets. Recent empirical results emphasize the role of noise traders in financial markets (Lee, Jiang, & Indro, 2002).2 Other work studies the influence of the presence of noise traders on asset prices. For example, De Long, Summers, and Waldmann (1990) suggest that a significant amount of volatility in stock prices is due to the unpredictability of noise traders’ beliefs. Daniel, Hirshleifer, and Subrahmanyam (1998) propose a theory of security market under- and over-reaction based on two well-known psychological biases. The first bias results from investor overconfidence about the precision of private information. The second bias results from self-attribution, which causes asymmetric shifts in investors’ confidence as a function of their investment outcomes. All of these studies use a similar definition of noise traders as those with erroneous stochastic beliefs, who falsely believe that they have special information about the future price of risky assets. These noise traders are different from Easley and O’Hara’s (1992) definition of noise traders, who falsely believe they have special information, to extend Easley and O’Hara’s (1992) seminal model. Our extended model demonstrates the existence of noise traders in the market narrows bid-ask spreads and slows down the speed of price reversion to the fundamental value. Furthermore, the bid-ask spread widens when noise trader sentiment aligns with the market maker’s prior beliefs. We show that the market maker’s ability to accurately predict noise traders’ sentiment is positively related to the quoted bid-ask spread and to the speed of price reversion. We demonstrate that Easley and O’Hara’s model is a special case of our model. Their conclusion that time is a factor in the security price adjustment process is strengthened in the presence of the erroneous sentiment of noise traders.

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uninformed traders. The noise traders’ trading activity is based on real information, albeit with sentiment noise. On the other hand, trading by Easley and O’Hara’s uninformed traders has no correlation with any relevant information.3

The importance of studying the behavior of noise traders is well established in the literature. However, the inclusion of noise traders in the extant market-microstructure literature, studying the dynamics of prices and the bid-ask spread through time, is scarce. To our best knowledge, only two papers follow that venue, to some extent. Wang (2002) extends Kyle (1985) who models the trading process in a rational-expectation framework in which orders are aggregated and batched by the market maker and are cleared at a single price. Wang’s definition of noise traders follows Black (1986). Noise traders, under this definition, believe they possess private information, however they only have noise. Under Black’s definition, noise traders’ beliefs are unbiased. In other words, they are not bullish or bearish over time, and thus on average they predict the true signal with accuracy. Wang’s framework yields a single clearing price in the market.

The model we derive below introduces noise traders to Easley and O’Hara’s (1992) trading mechanism. An important output of Easley and O’Hara’s model is the bid–ask spread. Thus, unlike Wang’s (2002) model, our model facilitates study of the influence of noise traders, their sentiment and its predictability on the bid–ask spread. Wang’s focus is on price efficiency, market liquidity, trading volume and traders’ profit.

Furthermore, we also focus on the influence of noise trading on the path of the price-adjustment process. This requires a dynamic model such as the Easley and O’Hara (1992) model that studies how prices are set and how they adjust over time. Hence, our focus is on the impact of noise trading on the market maker’s learning process and the speed of the price adjustment, as well as on the effect of noise traders’ sentiment and its predictability on this process. Thus, the focus of our paper is on different issues relative to that of Wang. Although his approach provides a number of insights into market phenomena, it is not amenable to the issues we study in our paper.

The second paper that includes noise traders in a model studying liquidity is Weston (2001). Similar to Wang (2002), Weston also uses Black’s (1986) definition of a noise trader, but he extends the work of Easley, Kiefer, O’Hara, and Paperman (1996). His empirical model is designed to facilitate an empirical investigation of the proportion of noise traders in different markets. Furthermore, his study is limited to the empirical study of noise traders’ influence on the relationship between informed trading and the bid–ask spread. Like Wang (2002), Weston’s model does not provide insight into the impact of noise traders on the market maker’s learning process, the price-adjustment process, and the formation of the bid–ask spread.

The current paper contributes to the extant literature by providing insight into the influence of the presence of noise traders on asset mispricing, the speed of price adjustment, and the relationship between liquidity and the process of price reversion. The paper better approximates reality by considering three types of traders that exist in most financial markets (for evidence see Weston, 2001); fully informed traders, noise traders, and liquidity traders. All propositions from Easley and O’Hara (1992) still hold in the context of our model. In addition, our model yields results, which are generated exclusively by the presence of noise traders in our setup.

Our model suggests that the existence of noise traders in the market narrows bid–ask spreads and slows down the speed of price reversion to the fundamental value. Furthermore, the market maker sets a wider bid–ask when noise traders’ sentiment aligns with his/her prior beliefs. We show that the market maker’s ability to accurately predict noise traders’ sentiment is positively related to the quoted bid–ask spread and to the speed of price reversion. Finally, we demonstrate that Easley and O’Hara’s model is a special case of our model. Thus, their conclusion that time is a factor in the security price adjustment process is also valid in our model, with stronger implications in the presence of noise traders’ erroneous sentiment.

The remainder of this paper is organized as follows: Section 2 discusses the characterization of noise traders in De Long et al. (1990). In Section 3 we present our general model that incorporates sophisticated (fully informed) traders, liquidity traders, and noise traders. Section 4 describes how the market maker sets bid and ask quotes, and how these quotes evolve throughout the trading day. In Section 5 we demonstrate how our extended model advances understanding of the security-price adjustment process and other phenomena in the market microstructure literature when noise traders are present. Summary and conclusions are offered in Section 6.

2. Noise traders

De Long et al. (1990) present a model that explains several financial anomalies, such as the excess volatility of asset prices, the mean reversion of stock returns, and the underpricing of close-end mutual funds. In their model, sophisticated traders are defined as investors who are capable of accurately perceiving the distribution of returns from holding a risky asset, and therefore accurately estimate its price. Noise traders, on the other hand, falsely believe they have special information about the future price of the risky asset. “They may get their pseudo signals from technical analysis, stock brokers, or economic consultants and irrationally believe that these signals carry information. Or in formulating their investment strategies, they may exhibit the fallacy of extra subjective certainty that has been repeatedly demonstrated in experimental contexts since Alpert and Raiffa (1982)” (De Long et al., 1990, p. 706). In other words, a noise trader misperceives the asset price.

De Long et al. assume that noise traders falsely believe that the price of the risky asset is $\rho$ dollars above its true value. They further assume that this dollar price misperception, $\rho$, is normally distributed. This implies that noise traders believe in the possibility of a negative asset value. To see this, let $\mathcal{V}$ denote the value of the risky asset under the “low signal” state, then for this state a noise trader falsely sets a value of $\mathcal{V} + \rho$ for the asset. Since under the normal distribution $\rho$ can take (infinitely) negative values, noise traders will attach a negative value for the asset under the “low signal” state whenever $\rho < -\mathcal{V}$.

To ensure that the noise trader always sets a nonnegative expected security price, we assume that $\ln(\rho + \mathcal{V})$ is normally distributed. This implies that at the minimum the (uncertain) dollar pricing error will take the value of: $\rho_{\text{min}} = -\mathcal{V}$. Thus, we assume that:

$$\ln(\rho + \mathcal{V}) \sim N(\rho^*, \sigma^2),$$

where $\rho^*$ is a measure of the mean price misperception. It indicates the average “bullishness” of noise traders resulting from their erroneous sentiment. $\sigma^2$ denotes the variance of noise traders’ price misperception.4 This variance measures the predictability of the noise trader’s sentiment. Recall that according to Black’s (1986) definition noise traders’ beliefs are unbiased, i.e., when using Black’s definition,
one implicitly assumes \( \rho^* = 0 \). We prefer our modification of De Long et al.’s definition since in reality noise traders’ erroneous sentiment can persist over time.

De Long et al. find the noise traders’ demand for risky assets by maximizing their expected utility given that they falsely believe that the distribution of the price of the risky asset next period is \( \rho \) dollars above its true value. They demonstrate several alternative ways of specifying noise traders’ demand that leads to the same equilibrium result. These specifications include noise traders that: (a) fix an asset price at which they buy or sell; (b) purchase a fixed quantity of the risky asset; and (c) under- or over-estimate the variance of returns. In our model, we choose to use alternative (a), which is the most suitable for our modeling environment.

3. A general model for price adjustment with noise traders

The process by which prices come to impound new information is a central issue in the field of market microstructure. To demonstrate the impact of noise traders on this security price-adjustment process, we first derive a general model in the spirit of Easley and O’Hara’s (1992) seminal work, which includes sophisticated (informed) traders, liquidity traders, and noise traders. The following discussion reviews our modeling environment and its relevant features, including player characterization, asset value, participation by traders and implicit trading day activity.

3.1. Player characterization

In this model, we have four player categories: a market maker, sophisticated (fully informed) traders, noise traders, and liquidity traders. The risk-neutral market maker is responsible for quoting buy and sell prices to potential traders who are risk-neutral and price-takers. Sophisticated traders have the same characteristics as the informed traders in Easley and O’Hara’s (1992) model. Similarly, liquidity traders are Easley and O’Hara’s uninformed traders. We add a new player category—noise traders—who are risk neutral, price-takers and have De Long et al.’s (1990) type perception errors in their beliefs.

3.2. Asset value

\( V \) represents the eventual value of an asset. An information event is defined as the occurrence of a high signal (\( H \)), a low signal (\( L \)), or a “no signal” event (\( O \)). Let \( \psi \) denote the signal, \( \psi = (L, H, O) \). For sophisticated traders, \( E[V|\psi = L] = V \) and \( E[V|\psi = H] = \tilde{V} \). Following De Long et al., \( \rho \) denotes the noise traders’ price misperception. Noise traders falsely believe that the price of the risky asset is \( \rho \) dollars above its true value. Thus, they set up the following conditional expectation: \( E[V|\psi = L] = V \) and \( E[V|\psi = H] = \tilde{V} + \rho \). Given this specification, one can view sophisticated traders as a special case of noise traders with: \( \rho^* = \sigma_{\rho}^2 = 0 \). Thus, below we categorize sophisticated traders and noise traders as members of a principal group of information-based traders.

We consider three alternative methods by which noise traders’ misconceptions may be modeled. Misconception may exist with respect to: (a) the high and low eventual values of the risky asset, \( V \) or \( \tilde{V} \); (b) the type of signal received, \( L, H, \) or \( O \); and (c) the expected value of the risky asset. Of these three methods, we choose the first one because it is intuitively appealing and most closely reflects the technology from De Long et al.’s model.

3.3. Traders’ participation

We adopt the simplifying convention of an exogenous arrival process, whose parameters correspond to a simple probabilistic structure, as in Glosten and Milgrom (1983) and Easley and O’Hara (1992). The inclusion of noise traders into Easley and O’Hara’s model changes traders’ participation in two ways. First, following the addition of noise traders, let \( \mu \) denote the market maker’s expectations with respect to the fraction of information-based trades. We denote the market maker’s expectation of the fraction of information-based traders who are sophisticated by \( \eta \).

Second, to significantly distinguish sophisticated traders from noise traders, the noise traders’ decision to purchase or sell the stock must not be a perfect function of the low or high signal. Instead, we allow noise traders to buy or sell after observing a high signal as well as after observing a low signal. Given this setup, information-based traders, which include sophisticated traders and noise traders, apply the following rule when deciding whether to purchase or sell an asset:

**An information-based trader purchases the asset when:**

\[
E[V|\psi = H] > V^* \text{ and sells when } E[V|\psi = L] < V^*.
\]

Hence, since \( V < V^* < \tilde{V} \) when \( \rho = 0 \), a sophisticated trader will always buy when the signal is \( H \) and sell when the signal is \( L \), as in Easley and O’Hara’s model. However, for a noise trader (\( \rho \neq 0 \)),\( V < V^* < \tilde{V} \) may not hold, i.e., when \( \rho = 0 \), there is a positive probability that an \( H \) signal will lead to a sell and an \( L \) signal will lead to a buy. The conditional probability of each of these events is a function of \( \rho \) and the first two moments of the distribution of \( \ln(\rho + V^*) \): \( \rho^* \) and \( \sigma_{\rho}^2 \).

Hence, we can solve for the values of these probabilities. Let \( \phi \) denote the conditional probability that, given the signal \( H \), the noise trader purchases the stock; \( \psi \) denote the conditional probability that, given the signal \( L \), the noise trader sells the stock. In Appendix A we show that, the assumed distribution for noise traders’ pricing error yields:

\[
\varphi = N \left\{ \frac{\ln(\rho + V)}{\sigma_{\rho};} \right\} \text{ and } \psi = N \left\{ \frac{\ln(\rho + V)}{\sigma_{\rho};} \right\}, \text{where } N [\cdot] \text{ denotes the cumulative distribution function for the standard normal distribution.}
\]

Based on this setup, we state traders’ participation as follows. First, to be consistent with Easley and O’Hara’s model, we let \( \gamma = 1 - \gamma; e^\phi \), \( \mu \) in our model denote the same parameters as in their model. A fraction \( \gamma \) of the uninformed (liquidity) traders are potential sellers and a fraction \( 1 - \gamma \) are potential buyers. At time \( t \), the probability of a trade by an uninformed potential buyer who checks a quote is \( e^{\phi}; \) if an uninformed seller checks a quote, the probability that \( s/he \) will sell is \( e^\psi \). If an information event occurs (with probability \( \alpha \)), the market maker expects the fraction of trades made by information-based traders to be \( \mu \).

3.4. Trading-day activity

We allow for a positive probability that any given trade is initiated by a noise trader and that the type of trade is not perfectly related to the signal. We incorporate this as follows. Trade may occur throughout the trading day. A single trading day is divided into discrete time intervals denoted by \( t = 1, 2, \ldots \). Each time interval is long enough to facilitate a maximum of one trade. Trading activity is sequential, with traders randomly selected to trade according to the above probabilities.

We focus on two-period activities. Prior to trading-day, nature decides the existence of a private information signal with probability \( \alpha \). If private information does exist, then \( Pr(\psi = L) = \delta \) and \( Pr(\psi = H) = 1 - \delta \). If no event took place before \( \psi = O \), only uninformed liquidity traders trade. A liquidity trader is either potential buyer or seller, and \( s/he \) either accepts or rejects a trade. If an event took place before, the market maker offers a trading opportunity to either information-based or liquidity traders. If the trader is information based and sophisticated, \( s/he \) buys or sells based on the signal; if the information-based trader is a noise trader, \( s/he \) may buy or sell with probabilities \( \phi \) and \( \psi \) (depending on the type of signal). If the trader is a liquidity trader, \( s/he \) trades as if no event took place.
The tree diagram in Fig. 1 outlines this trading structure, which is closely related to Easley and O’Hara’s (1992) time specification. At the first node, nature chooses whether an information event occurs. If private information exists, the type of signal (either H or L) is determined at the second node. These two nodes occur before and up to the beginning of the day. From this point on, traders are selected at each time \( t \) to trade based on the probability structure described above. Hence, in the event that a private signal exists, an information-based trader is selected with probability \( \mu \). Note that there is a probability \( \eta \) that this information-based trader is a sophisticated trader and a probability \( 1 - \eta \) that s/he is a noise trader. In the same way, there is a probability of \( 1 - \mu \) that the trader is a liquidity trader. For a trade in the following time interval, the game progresses from the right of the dotted line in the diagram, but only the trader selection process is repeated, and continues throughout the day. Note that the trading structure in Easley and O’Hara’s (1992) model is a special case of our trading structure (with \( \eta = 1 \)). Thus, one can easily show that Easley and O’Hara’s model is a special case of our more general model.5

Fig. 1. The trading structure in a market with sophisticated, noise and liquidity traders. \( \alpha \) denotes the probability of an information event, \( \delta \) denotes the probability of a low signal, \( \mu \) denotes the probability that the trade originates from a sophisticated (fully-informed) trader, \( \gamma \) denotes the fraction of liquidity traders who are potential sellers, and \( \epsilon^S(\epsilon^B) \) denotes the probability that the liquidity trader will actually trade. \( \eta \) is the fraction of information-based traders who are sophisticated, \( \phi \) is the probability that a noise trader will buy under a high signal, \( \phi \) is the probability that a noise trader will sell under a low signal. Nodes to the left of the dotted line take place only at the beginning of the trading day, while nodes to the right are possible at each trading interval.

5 We also verified that all propositions presented by Easley and O’Hara still hold in the context of our model.
To see why the market maker prefers liquidity traders to noise traders, consider Table 1 that outlines the probabilities assigned by the market maker to the actions taken by noise traders and liquidity traders for the two types of signals, together with the following lemma.

**Lemma 1.** The sum of the conditional probabilities of buy high and sell low is greater than 1 for the noise trader.

**Proof.** All proofs of lemmas are in Appendix B.

Lemma 1 shows that for the noise trader (Panel A) the sum of the conditional probabilities of buy high (\(\varphi\)) and sell low (\(\varphi^*\)) is greater than 1. At the same time, for the liquidity trader (Panel B) the sum of these probabilities (\((1 - \gamma)e^B\) and \(\gamma e^B\), respectively) is lower than or equal to 1, by definition. This implies a higher chance that liquidity traders will err in their actions relative to noise traders.

In the context of a trinomial tree, any given change in the proportion of noise traders results in a change in the arrival rates of both sophisticated traders and liquidity traders. Since the market maker clearly prefers that the proportion of liquidity traders increase at the expense of sophisticated traders, with a trinomial tree one cannot analytically isolate the influence of noise traders. Thus, in our model we treat both sophisticated traders and noise traders as information-based traders, and liquidity traders as uninformed traders. Since noise traders trade based on the signal, it can be shown that sophisticated traders are actually noise traders with: \(\rho^* = c^B \varphi^* = 0\). On the other hand, liquidity traders engage only in liquidity-motivated trades that are not driven by information and their demand is exogenous to the model. This structure is consistent evidence presented by Weston (2001). Weston empirically measures the proportion of noise trading in both NYSE and NASDAQ and finds that the introduction of noise trading greatly reduces the proportion of informed trading, but leaves the proportion of liquidity trading unchanged.

### 4. The evolution of quotes

The next issue to be stressed in this market structure is how prices evolve throughout the day. We focus on outlining the quote-setting process with noise traders and then make comparisons with models that exclude noise traders. We start from the first trade of the day. To set his/her initial price quotes, the market maker calculates the expected value of the asset conditioned on the type of trade that may occur. To this end, one must first obtain the probability of a low value, conditional on the type of trade: \(\text{Pr}(V = V | Q, Q = [B, S, N])\) (\(B = \text{buy}, S = \text{sell}, N = \text{no trade}\)). If a low signal occurs, the market maker sets \(\text{Pr}(V = V) = 1\), if a high signal occurs s/he sets \(\text{Pr}(V = V) = 0\), and if no signal occurs, the market maker sets \(\text{Pr}(V = V) = \delta\). Since the market maker can only observe trade outcome, \(Q = [B, S, N]\), the updating formula for the conditional probability is given by:

\[
\tilde{\varphi}(Q) = \text{Pr}(V = V | Q) = 1 - \text{Pr}(\psi = L | Q) + \delta \times \text{Pr}(\psi = O | Q).
\]

Since the market maker is Bayesian, then following the first trade Bayes rule gives these conditional probabilities:

\[
\text{Pr}(\psi = X | Q) = \frac{\text{Pr}(\psi = X) \text{Pr}(X | Q)}{\text{Pr}(\psi = H | Q) + \text{Pr}(\psi = O | Q) \text{Pr}(Q | O)}.
\]

To calculate \(\text{Pr}(\psi = L | S)\), the value of \(\delta\) given that the first trade was a sale, we note that

\[
\text{Pr}(\psi = L) = \alpha \delta,
\]

\[
\text{Pr}(\psi = H) = \alpha (1 - \delta),
\]

\[
\text{Pr}(\psi = O) = 1 - \alpha.
\]

Thus, the updating formula given that the first trade was a sale is given by:

\[
\tilde{\varphi}(S) = \text{Pr}(V = V | S) = 1 - \text{Pr}(\psi = L | S) + \delta \times \text{Pr}(\psi = O | S),
\]

\[
\tilde{\varphi}(S) = \delta \text{Pr}(\psi = L | S) + \delta \text{Pr}(\psi = O | S).
\]

Similarly, we can calculate \(\text{Pr}(\psi = L | B)\) and the updating formula given that the first trade was a buy:

\[
\tilde{\varphi}(B) = \text{Pr}(V = V | B) = 1 - \text{Pr}(\psi = L | B) + \delta \times \text{Pr}(\psi = O | B),
\]

\[
\tilde{\varphi}(B) = \delta \text{Pr}(\psi = L | B) + \delta \text{Pr}(\psi = O | B).
\]

Hence, the initial bid and (\(b_1\)) ask (\(a_1\)) prices are given by:

\[
b_1 = E[V/S] = \tilde{\varphi}(S) \frac{V}{\tilde{\varphi}(S)} + (1 - \tilde{\varphi}(S)) V,
\]

and

\[
a_1 = E[V/B] = \tilde{\varphi}(B) \frac{V}{\tilde{\varphi}(B)} + (1 - \tilde{\varphi}(B)) V.
\]

Let SP denote the bid-ask spread, then Eqs. (3) and (4) imply that the initial spread is given by:

\[
SP_1 = a_1 - b_1 = (\tilde{\varphi}(S) - \tilde{\varphi}(B)) (V - V).
\]

So far, we have described the structure of the model and determined initial equilibrium quotes. Before turning to the price-adjustment process, we discuss the properties of the above conditional probabilities. Easley and O’Hara (1992) show that without noise
traders these probabilities exhibit: $\delta(S) > 0$ and $\delta(B) < 0$. We show below that this is still the case with the addition of noise traders.

**Lemma 2.** The market maker increases the probability s/he attaches to $V$, given that a trader places a sell order; and does the opposite given that a trader places a buy order.

Lemma 2 shows that, in the presence of noise traders, the market maker adjusts his/her belief according to the trading outcome in the same manner as s/he does in the absence of noise traders.

Next, we examine how quotes evolve in a market where noise traders are present. At any time $t$, the trading outcome is given by $V \in \{B_t, S_t, N_t\}$. By Bayes rule, the market maker’s beliefs at the beginning of period $t$ are based on the history of trade. Let $\theta_{hi}$ denote the conditional probability that no signal has occurred, $\theta_{hi}$ denote the conditional probability that a high signal has occurred, and $\theta_{hi}$ denote the conditional probability that a low signal has occurred, such that:

$$\theta_{hi} = \Pr(\psi = O|Q^{-1})$$
$$\theta_{hi} = \Pr(\psi = H|Q^{-1})$$
$$\theta_{hi} = \Pr(\psi = L|Q^{-1})$$

Then the market maker’s bid quote for period $t$ is given by the expected value of $V$ conditional on the history of trade and a sale order at time $t$:

$$b_t = E[V|Q^{-1}, S] = \Pr(\psi = L|Q^{-1}, S)V + \Pr(\psi = H|Q^{-1}, S)V$$

+ $\Pr(\psi = O|Q^{-1}, S)V$.

Similarly, the market maker’s ask quote for period $t$ is given by:

$$a_t = E[V|Q^{-1}, B] = \Pr(\psi = L|Q^{-1}, B)V + \Pr(\psi = H|Q^{-1}, B)V$$

+ $\Pr(\psi = O|Q^{-1}, B)V$.

The above equations imply that we can obtain $a_t$ and $b_t$ if and only if we know the market maker’s beliefs (the conditional probabilities). Suppose that in the past $t$ trading intervals, the market maker has observed $n_t$ no trades, $b_t$ buys and $s_t$ sells, then

$$Pr(\psi = O|Q^{-1}) = Pr(\psi = O|O)Pr(O|Q^{-1})$$

$$Pr(\psi = O) = (\gamma e^\beta)^{(1-\gamma e^\beta)^{1/(1-\gamma e^\beta)}}(1-(1-\gamma e^\beta)^{1/(1-\gamma e^\beta)})$$

$$Pr(\psi = H|Q^{-1}) = \mu(1-\gamma e^\beta)^{\mu(1-\gamma e^\beta)^{1/(1-\gamma e^\beta)}}(1-(1-\gamma e^\beta)^{1/(1-\gamma e^\beta)})$$

$$Pr(\psi = L|Q^{-1}) = \mu(1-\gamma e^\beta)^{\mu(1-\gamma e^\beta)^{1/(1-\gamma e^\beta)}}(1-(1-\gamma e^\beta)^{1/(1-\gamma e^\beta)})$$

Thus, the market maker's conditional probability is given by:

$$Pr(\psi = O|Q^{-1}, S) = \mu(1-\gamma e^\beta)^{\mu(1-\gamma e^\beta)^{1/(1-\gamma e^\beta)}}(1-(1-\gamma e^\beta)^{1/(1-\gamma e^\beta)})$$

$$+ \mu(1-\gamma e^\beta)^{\mu(1-\gamma e^\beta)^{1/(1-\gamma e^\beta)}}(1-(1-\gamma e^\beta)^{1/(1-\gamma e^\beta)})$$

$$+ \mu(1-\gamma e^\beta)^{\mu(1-\gamma e^\beta)^{1/(1-\gamma e^\beta)}}(1-(1-\gamma e^\beta)^{1/(1-\gamma e^\beta)})$$

(5)

The market maker’s quotes for period $t + 1$ can be written as

$$b_{t+1} = E[V|Q^t, S] = Pr(\psi = L|n_t, s_t + 1, \beta)\bar{V} + Pr(\psi = H|n_t, s_t$$

$$+ 1, \beta)\bar{V} + Pr(\psi = O|n_t, s_t + 1, \beta)\bar{V}.$$

(6)

and

$$a_{t+1} = E[V|Q^t, B] = Pr(\psi = L|n_t, s_t + 1, \beta)\bar{V} + Pr(\psi = H|n_t, s_t$$

$$+ 1, \beta)\bar{V} + Pr(\psi = O|n_t, s_t + 1, \beta)\bar{V}.$$

(7)

From the above equations, we can see that quotes at time $t + 1$ depend not only on the latest trade, but also on previous trading activity $(n_t, s_t, \beta_t)$. The trading volume until time $t$ is given by $v_t = s_t + b_t$, and the market maker’s inventory position at time $t$ is given by $i_t = s_t - b_t$. Since during every time period, a trade may or may not take place, the number of periods must equal $n_t + b_t + s_t$. In our model, if one knows $v_t$ and $b_t$, one can compute $b_t$ and $s_t$; but since quotes depend also on $n_t$, we also need to know the number of periods to estimate quotes. Thus, like in Easley and O’Hara’s (1992), time per se plays a role in our model with noise traders as it appears in the security price-adjustment process. Moreover, in agreement with Easley and O’Hara’s model, our model predicts that volume is also a factor in the formation of the bid-ask spread.

5. Do sentimental noise traders matter?

In this section we focus on comparing the implications of our model to the special case where noise traders do not exist as in Easley and O’Hara (1992). We discuss the influence of noise traders on the bid-ask spread and the security price-adjustment process. As mentioned before, we verified that all propositions presented by Easley and O’Hara still hold in the context of our model. In addition, our model yields the following results, which are generated exclusively by the presence of noise traders in our setup.

5.1. Noise trading and the bid-ask spread

**Proposition 1.** Bid-ask spreads narrow in the presence of noise traders.

**Proof.** All proofs of propositions are in Appendix B.

**Proposition 2.** The higher the fraction of noise traders the lower the bid-ask spread.

The intuition behind Propositions 1 and 2 is that a higher noise-trading activity reduces the information asymmetry that the market maker faces when facing informed traders. Facing lower risk, the market maker sets a narrower bid-ask spread.

**Proposition 3.** The greater the uncertainty regarding noise traders’ sentiment ($\sigma^2$), the narrower the bid-ask spread.

This proposition implies that as noise traders’ sentiment becomes more imprecise, their trading will result in a lower exposure to asymmetric-information risk for the market maker, and so he sets a lower bid-ask spread. Increased vagueness about noise trader’s sentiment implies a lower probability perceived by the market maker that a trade is based on information. As the variance of the noise increases, noise traders’ behavior is perceived as more erratic, and their trade decisions seem less related to the observed signal. Thus, from the market maker’s point of view, an increase in the variance of price misperception makes the noise traders look more like liquidity traders who trade randomly, independent of any relevant information. On the other hand, when the noise traders’ sentiment is perfectly clear ($\sigma^2 = 0$), the market maker perceives their market activity to be linear in the signal they observe.

In our model, noise traders’ irrationalpricing errors are represented by $\rho^\ast$. Easterwood and Nutt’s (1999) empirical study provides evidence that traders tend to underreact to negative information and overreact to positive information. This implies that investors are on average optimistic, and $\rho^\ast > 0$. Although this result may hold on average, assuming a positive $\rho^\ast$ does not accurately model real markets, as there are instances where traders’ pessimistic reaction implies a negative pricing error. Consequently, in our model we make no assumptions with respect to the sign of $\rho^\ast$. Thus, our model allows additional insight into the effects of investors’ sentiment, being either optimistic or pessimistic. This is reflected in the following proposition.
Proposition 4. If uninformed traders are equally likely to buy or sell, ceteris paribus, bid-ask spreads will be wider when the market maker’s beliefs and noise traders’ sentiment align than otherwise. Specifically, when the market maker perceives a higher (lower) probability of low signal, noise traders’ pessimistic (optimistic) sentiments will result in a wider bid-ask spread than the case in which noise traders’ sentiment and the market maker’s perceptions differ.

We view this proposition as the most intriguing result of the current model. Our model predicts that bid-ask spread oscillates with noise traders’ optimistic and pessimistic sentiments conditional on the market maker’s private beliefs. Our model adds to the extant literature on the formation of the bid-ask spread, by showing that the interaction between noise traders’ sentiment and the market maker’s belief is also a determinant of the spread.

To see the intuition behind this result, suppose that the market maker has conjectured that it is very likely a low signal has happened. At the same time s/he predicts that noise traders’ sentiment is pessimistic. In other words, the market maker believes that noise traders are likely to make correct sell decisions, similar to informed traders. This implies higher adverse selection costs for the market maker, who sets a wider bid-ask spread. Conversely, suppose that instead the market maker perceives a disagreement with pessimistic noise traders’ sentiment. This implies that the market maker believes that noise traders are likely to make wrong buy and sell decisions, unlike informed traders. In other words, in this case the adverse-selection cost is low for the market maker, who sets a narrower bid-ask spread.

This result is intuitive and has strong empirical implications. Until now, many researchers have studied investors’ sentiment in relation to market efficiency, price volatility, and other anomalies. To our knowledge, there are very limited studies in the extant literature looking at the influence of investors’ sentiments (bullishness and bearishness) on market bid-ask spreads. A recent paper by Chen, Christine, Kim, and McInish (2003) investigates closed-end funds and finds an inverse relationship between adverse-selection costs and investors’ sentiment. Thus, our model provides a more complete picture of the dynamics of the bid-ask spread.

5.2. Noise trading and the speed of price reversion

We adopt Easley and O’Hara’s (1992) approach to determine how quickly prices (approximately) reach their strong-form efficient levels. Our emphasis is on calculating the actual rate of convergence of the stochastic process, and then using it as a benchmark in comparing how various parameters affect the rate of convergence. Easley and O’Hara show that prices converge almost surely to the correct asset value. Here, we are interested in what influence the presence of noise traders might have on the speed of this price adjustment.

As quotes, and thus transaction prices, are linear in beliefs, the rate of convergence of price is the same as the rate of convergence of beliefs. For each signal \( \psi = [H, L, O] \) define the probability on trades as \( P^\psi = (P^\psi(N), P^\psi(B), P^\psi(S)) \), where a representative element, \( P^\psi(Q) \), is the probability of trade Q given a signal \( \psi \). Following Easley and O’Hara’s (1992), for any two probabilities on trades, \( P^\psi \) and \( P^\psi_0 \), we define the entropy of \( P^\psi_0 \) to \( P^\psi \) by

\[
I_p^\psi(P^\psi_0) = \sum_{Q=\{N, B, S\}} P^\psi_0(Q)\log\frac{P^\psi(Q)}{P^\psi_0(Q)}, \quad Q = \{N, B, S\}.
\]

Proposition 5. Transaction prices converge almost surely to their strong form efficient value at an exponential rate. If a signal \( \psi \) takes place, then the exponential rate is \( r(\psi) \). Since there are three possibilities of signal \( \psi \), we define \( r(\psi) = \min\{r^N(\psi), r^B(\psi), r^S(\psi): \psi \neq \emptyset\} \).

Proposition 6. These exponential rates of convergence decrease in both the fraction of trades from the noise-trader group and the variance of the noise when liquidity traders are equally likely to buy or sell.

Proposition 7. These rates of convergence can increase or decrease in the noise traders’ sentiment depending on which signal has materialized. Specifically, given that a high (low) signal has materialized, the rate of convergence will be higher if noise traders are overly pessimistic (optimistic) than when they are overly optimistic (pessimistic).

Propositions 5 through 7 provide insight into how noise traders influence the price-adjustment process. If we use the speed of price reversion as a measure of market inefficiency, Proposition 6 clearly points out that a higher participation of noise traders will result in an increasingly inefficient market. Stated differently, it takes a longer time for the market maker to reach strong-form efficiency price levels. Increasing the fraction of trades from the noise-trader group slows down the price-adjustment process, because their trading activity is based on noise as opposed to the true underlying signal. This echoes Ratches’s (1999) empirical study, which shows that arbitrageurs appear to be limited in their ability to eliminate deviations from fundamentals in the presence of noise traders.

Proposition 6 may also be important in the context of short-term contrarian strategies. For example, Jegadeesh and Titman (1995) demonstrate the existence of short-term contrarian profits. They attribute these abnormal profits to investors’ overreaction to firm-specific information. Their results suggest that asset prices evolve in a cyclical manner. Proposition 6 implies that, as noise traders participate in the market increases, these cycles tend to be longer, due to the longer reversion process of asset prices towards their fundamental values.

Proposition 6 further implies that a lower variance of noise, i.e., an improved ability of the market maker to predict noise traders’ sentiment, hastens the price-adjustment process, and increases the efficiency in which prices adjust to fundamentals. This is because as the variance of the noise decreases, noise traders’ behavior is perceived to be less erratic, and their trade decisions seem more related to the observed signal. When the noise traders’ sentiment is perfectly clear (\( \sigma^2 = 0 \)), the market maker perceives their market activity to be linear in the signal they observe, and to reveal the underlying information faster.

Proposition 7 points out that the relationship between the speed of the price adjustment and the noise traders’ sentiment depends on what signal has materialized. If noise traders are pessimistic, they tend to sell. If a low signal has occurred, sophisticated traders also sell. In this case it is difficult for the market maker to conjecture whether the order is based on information or just noise. By contrast, if a high signal has materialized, sophisticated traders will buy. In this case, the market maker will tend to view buy orders as potentially from informed traders, and s/he will adjust quotes faster to the fundamental value.

6. Summary and conclusions

Numerous studies in the extant literature investigate the dynamic process by which prices come to impound information over time (see a survey by Madhavan, 2000). We develop a new general model, based on Easley and O’Hara’s (1992) seminal work, to further our understanding of such a dynamic price adjustment process in the presence of noise traders. Other researchers refer to noise traders in different contexts (Glosten & Milgrom, 1985; Kyle, 1985; Lee, Mucklow, & Ready, 1993). These studies define noise traders as investors with no private information to exploit. We believe that such a definition of noise traders inevitably oversimplifies the problem.

Since De Long et al. (1990) work on noise-traders’ sentiment, tremendous evidence has accumulated in the literature showing that noise traders influence security prices to a large degree. We follow De Long et al’s definition of noise traders who falsely believe they have special information and act on such information. We present such noise traders into Easley and O’Hara’s (1992) model, and demonstrate their influence on the formation of bid-ask spreads.
and the speed of price adjustment. This is a novel approach insofar as it has not been considered in either standard noise-trader models or standard price-discovery models. We verified that all propositions presented by Easley and O’Hara still hold in the context of our model. In addition, our model yields results, which are generated exclusively by the presence of noise traders in our setup.

We demonstrate that noise-trading activity reduces the risk of asymmetric information that the market maker faces when trading with informed traders, and as a result the market maker sets a narrower bid-ask spread. Furthermore, our model suggests that as the market maker’s ability to predict noise traders’ sentiment diminishes, the market maker perceives a lower level of exposure to asymmetric-information risk. As a result, s/he sets a lower bid-ask spread. Our research also demonstrates that, ceteris paribus, when noise traders’ sentiment aligns with the market maker’s prior beliefs, the market maker believes that noise traders are likely to make correct buy and sell decisions, and s/he quotes a wider bid-ask spread. Finally, our model predicts that a higher participation of noise traders, as well as the market maker’s ability to predict noise traders’ sentiment, results in a slower price-adjustment process, and an increasingly inefficient market. We also show that the relationship between the speed of the price adjustment and the noise traders’ sentiment depends on what signal has materialized. The above predictions of our model have several empirical implications that are left for future research.

Appendix A. The derivation of $\phi$ and $\varphi$

Let $\phi$ denote the probability that, given the signal $H$, the noise trader purchases the stock, and $\varphi$ denote the probability that, given the signal $L$, the noise trader sells the stock. The trading rule for information-based traders (sophisticated traders and noise traders) is:

An information-based trader purchases the stock when $E[\psi | H] > V^*$ and sells when $E[\psi | L] < V^*$. Therefore, for noise traders, we have the following conditional probabilities:

$$\phi = Pr[\psi > \delta V + (1 - \delta) V] \quad \text{and} \quad \varphi = Pr[\psi > \delta V + (1 - \delta) V].$$

To ensure that the noise trader always sets a nonnegative expected price, we assume that $\ln(p + \psi) - N(p^e \sigma_p^e)$, so that at the minimum the (uncertain) dollar pricing error will take the value of: $p_{\min} = -V$. Let $x = p + \psi$, the following is the probability density function of $x$:

$$f(x) = \frac{1}{\sqrt{2\pi} \sigma_p} x \exp \left\{ -\frac{1}{2} \left( \frac{x - p^e}{\sigma_p} \right)^2 \right\}.$$  

Thus, the above conditional probabilities are given by:

$$\varphi = Pr[x > V - (\delta - V)] = \Phi \left( \frac{\ln(V) - \mu - \ln(\delta - V)}{\sigma} \right),$$

and

$$\phi = Pr[x < V] = \frac{N\left( \ln(V) - p^e \sigma_p \right)}{\sigma_p},$$

where $N[·]$ denotes the cumulative distribution function for the standard normal distribution.

Appendix B. Proofs of lemmas propositions

Proof of Lemma 1. We need to show that $(\varphi + \varphi - 1) > 0$.

Substituting $\varphi = N\left( \frac{\ln(V) - \mu - \ln(\delta - V)}{\sigma} \right)$, and $\phi = N\left( \ln(V) - p^e \sigma_p \right)$, we get:

$$N\left( \frac{\ln(V) - \mu - \ln(\delta - V)}{\sigma_p} \right) + N\left( \ln(V) - p^e \sigma_p \right) - 1 > 0.$$  

Simplifying, we get:

$$N\left( \frac{\ln(V) - \mu - \ln(\delta - V)}{\sigma_p} \right) > N\left( \frac{\ln(V) - \mu}{\sigma_p} \right).$$

This inequality must hold since the value in the cumulative distribution function on the left-hand side is greater than the value in the cumulative distribution function on the right-hand side. Q.E.D.

Proof of Lemma 2. To show that $\delta(S) > \delta$ in the presence of noise traders, according to Eq. (1) we need to show that:

$$\frac{\varphi(S)}{\varphi(S)} + (1 - \varphi(S)) \gamma_b^2 > \frac{\varphi(S)}{\varphi(S)} + (1 - \varphi(S)) \gamma_b^2.$$  

Simplifying this expression, we get:

$$\eta > 1 - \gamma_b S > 0.$$  

In Lemma 1 we show that $(\varphi + \varphi - 1) > 0$. Since $0 < \eta < 1$ and $\eta < 1$, we must have: $\delta(S) > \delta$.

Similarly, to show that $\delta(B) > \delta$ with noise traders, according to Eq. (2) we need to show that:

$$\frac{\varphi(B)}{\varphi(B)} (1 - \varphi(B)) + (1 - \varphi(B)) \gamma_b^2 > \frac{\varphi(B)}{\varphi(B)} (1 - \varphi(B)) + (1 - \varphi(B)) \gamma_b^2.$$  

Simplifying this expression, we get: $\eta > 1 - \varphi(S) > 0$, which is the same inequality as in the above case. Thus, we also have $\delta(B) > \delta$. Q.E.D.

Proof of Proposition 1. Eqs. (3) and (4) show that quotes are linear combinations of the market maker’s beliefs. Thus, to examine the impact of the presence of noise traders on the bid-ask spread, we need to examine the conditional beliefs: $\delta(S) > \delta(S) = \delta(S) > 1$, and $\delta(B) > \delta(B) > 1$, where $\delta(S)$ and $\delta(B)$ are the conditional beliefs in Easley and O’Hara’s (1992) model. These beliefs are given in the special case of the current model, when noise traders are absent ($\eta = 1$).

First, we show that $\delta(S) > \delta(S)$. Substituting for $\delta(S)$ and $\delta(S)$ from Eq. (1), this inequality becomes:

$$\delta \varphi(S) + (1 - \varphi(S)) \gamma_b^2 > \frac{\varphi(S)}{\varphi(S)} + (1 - \varphi(S)) \gamma_b^2.$$  

Simplifying, we get: $\varphi(S) > (1 - \varphi(S)) \gamma_b^2 > 0$. Since all symbols in this inequality represent probabilities, and therefore have values between 0 and 1, the above inequality must hold. Thus, we prove that $\delta(S) > \delta(S)$.

Next, we show that $\delta(B) > \delta(B)$. Substituting for $\delta(B)$ and $\delta(B)$ from Eq. (2), this inequality becomes:

$$\delta \varphi(B) + (1 - \varphi(B)) \gamma_b^2 > \frac{\varphi(B)}{\varphi(B)} + (1 - \varphi(B)) \gamma_b^2.$$  

Simplifying, we get: $\varphi(B) > (1 - \varphi(B)) \gamma_b^2 > 0$. Again, since these probabilities can only take values between 0 and 1, the above inequality must hold. Thus, we prove that $\delta(B) > \delta(B)$.

Finally, let $SP$ and $SP^a$ denote the initial spread with and without noise traders, respectively. Also, let $a^*$ and $b^*$ denote the bid and ask prices when noise traders are absent, respectively. Then, Eqs. (3) and (4) imply:

$$SP = a - b = (\varphi(b) - \delta(S))(b - \delta(B)),$$  

(A.1)
and
\[ SP' = a' - b' = (V - V') (\delta'(S) - \delta'(B)). \] (A.2)

Since \( \delta(S) < \delta(S)' \) and \( \delta(B) > \delta(B)' \), then Eqs. (A.1) and (A.2) imply: \( SP < SP' \).

**Proof of Proposition 2.** The fraction of noise traders is measured by \( 1 - \eta \). Since by Eqs. (3) and (4) quotes are linear combinations of the market maker’s beliefs, we need to examine how \( \eta \) affects \( \delta(S) \) and \( \delta(B) \).

One can easily show that the sign of \( \delta(S)/\partial \eta \) is identical to the sign of:
\[ (\alpha_\mu)^2 (1 - \delta)(1 - \phi) + \alpha_\mu (1 - \alpha_\mu) T_e^e (1 - \delta)(2 - \varphi - \phi). \]

Since all symbols in this expression represent probabilities, this expression must be positive, and therefore: \( \delta(S)/\partial \eta > 0 \). Similarly, one can show that \( \delta(B)/\partial \eta > 0 \). These results, together with Eq. (A.1), imply that: \( \delta SP/\partial \eta > 0 \). This means that as the fraction of noise traders \( (1 - \eta) \) increases, the bid-ask spread narrows. Q.E.D.

**Proof of Proposition 3.** By Eq. (A.1), to examine the relationship between the spread \( (SP) \) and the variance of the noise \( (\sigma_\eta) \), one needs to look at the following expression:
\[ \frac{\partial SP}{\partial \sigma_\eta} = \left( \frac{\partial \xi}{\partial \sigma_\eta} \right) \left( \frac{\partial \delta(S)}{\partial \sigma_{T_p}} + \frac{\partial \delta(S) \delta(B)}{\partial \sigma_{T_p}} \right) - \left( \frac{\partial \delta(B)}{\partial \sigma_{T_p}} \frac{\partial \delta(B)}{\partial \sigma_{T_p}} \right). \] (A.3)

Since \( \sigma_\eta \) does not directly appear in Eqs. (A.1) and (1) to (4), we need to look at its impact on both \( \phi \) and \( \varphi \). One can easily show that the sign of \( \delta(S)/\partial \sigma_\eta \) is the same as the sign of: \( \alpha_\mu (1 - \delta)(1 - \eta)(1 - \gamma) \alpha_\mu(1 - \gamma) + (1 - \gamma) T_e^e \), and the sign of \( \delta(B)/\partial \sigma_\eta \) is the same as the sign of: \( (\alpha_\mu)(1 - \eta)(1 - \delta)(1 - \phi) + (1 - \gamma) T_e^e (1 - \delta)(1 - \eta) \). Since all symbols in these expressions represent probabilities, we have: \( \delta(S)/\partial \sigma_\eta > 0 \) and \( \delta(S)/\partial \sigma_\eta > 0 \). Similarly we can show that \( \delta(B)/\partial \sigma_\eta > 0 \), and \( \delta(B)/\partial \sigma_\eta > 0 \).

Both \( \varphi \sim N \left[ \frac{\eta - \ln (\sigma^2 - V')}{\sigma_\varphi} \right] \) and \( \phi \sim N \left[ \frac{\ln (\sigma^2 - V')}{\sigma_\phi} \right] \) are cumulative distribution functions, and therefore both decrease in \( \sigma_\eta \). Thus, the signs of the above derivatives together with Eq. (A.3) imply that the relationship between uncertainty regarding noise traders' sentiment and the bid-ask spread is negative: \( \delta SP/\partial \sigma_\eta > 0 \).

**Proof of Proposition 4.** Let \( \delta(S)^0, \delta(B)^0; \delta(S)^0, \delta(B)^0 \) denote the market maker's subjective probabilities of a low signal, when noise traders are overpessimistic \( (P) \) and overoptimistic \( (O) \). \( SP^0 \) and \( SP^0 \) denote the bid-ask spread corresponding to pessimistic and optimistic sentiment, respectively. Consider the following two extreme cases:

(i) When the noise trader is highly optimistic \( (\varphi \rightarrow 0 \) ), the probability that, given the signal \( L \), the noise trader sells the stock is very low \( (\varphi \rightarrow 0) \), and the probability that, given the signal \( H \), the noise trader purchases the stock is very high \( (\varphi \rightarrow 1) \). According to this scenario of an optimistic noise trader, the market maker’s subjective probabilities of a low signal are given by:
\[ \delta(S)^0 = \delta \frac{\alpha_\mu \eta + (1 - \alpha_\mu) \gamma e^5}{\alpha_\mu \eta + (1 - \alpha_\mu) \gamma e^5}, \]
and
\[ \delta(B)^0 = \delta \frac{\alpha_\mu (1 - \eta) + (1 - \alpha_\mu)(1 - \gamma) e^8}{\alpha_\mu (1 - \eta) + (1 - \alpha_\mu)(1 - \gamma) e^8}. \]

The difference between the market maker's subjective probability of a low signal following a buy order and that following a sell order, is given by:
\[ \delta(S)^0 - \delta(B)^0 \]
\[ = \frac{(\alpha_\mu)^2 \gamma \eta (1 - \delta) + \alpha_\mu (1 - \alpha_\mu) (1 - \gamma) e^8 + \alpha_\mu (1 - \alpha_\mu) (1 - \delta) \gamma e^5}{\alpha_\mu \eta (1 - \delta) + (1 - \alpha_\mu)(1 - \gamma) e^8}. \] (A.4)

(ii) When the noise trader is highly pessimistic \( (\varphi \rightarrow \infty) \), the probability that, given the signal \( L \), the noise trader sells the stock is very high \( (\varphi \rightarrow 1) \), and the probability that, given the signal \( H \), the noise trader purchases the stock is very low \( (\varphi \rightarrow 0) \). Under this scenario of an overpessimistic noise trader, the market maker’s subjective probabilities of a low signal are given by:
\[ \delta(S)^0 = \delta \frac{\alpha_\mu (1 - \alpha_\mu) \gamma e^5}{\alpha_\mu (1 - \delta)(1 - \eta) + (1 - \alpha_\mu)(1 - \gamma) e^8}, \]
and
\[ \delta(B)^0 = \delta \frac{(1 - \alpha_\mu)(1 - \gamma) e^8}{\alpha_\mu (1 - \delta)(1 - \eta) + (1 - \alpha_\mu)(1 - \gamma) e^8}. \]

The difference between the market maker's subjective probability of a low signal following a buy order and that following a sell order, is given by:
\[ \delta(S)^0 - \delta(B)^0 \]
\[ = \frac{(\alpha_\mu)^2 \gamma \eta (1 - \delta) + \alpha_\mu (1 - \alpha_\mu) (1 - \gamma) e^8 + \alpha_\mu (1 - \alpha_\mu) (1 - \delta) \gamma e^5}{\alpha_\mu \eta (1 - \delta) + (1 - \alpha_\mu)(1 - \gamma) e^8}. \] (A.5)

Eqs. (A.1), (A.4), and (A.5) imply the following ratio: \( \frac{SP^0}{SP^0} = \frac{SP^0}{SP^0} = \frac{SP^0}{SP^0} \)
\[ = \frac{\alpha_\mu \eta (1 - \delta) + (1 - \alpha_\mu)(1 - \gamma) e^8}{\alpha_\mu \eta (1 - \delta) + (1 - \alpha_\mu)(1 - \gamma) e^8}. \]

We assume that uninformed traders are equally likely to buy or sell: \( ye^5 = (1 - \gamma) e^8 \). Setting: \( SP^0 > SP^0 \), we get \( \gamma(1 - \eta)/\alpha_\mu > 0, \) or \( \delta > 0.5, \) i.e., \( SP^0 > SP^0 \) if and only if \( \delta > 0.5 \). In other words, if the market maker is pessimistic \( (\delta > 0.5) \), s/he sets a higher spread when the noise trader is also optimistic \( (SP^0) \), than the spread s/he sets when the noise trader is pessimistic \( (SP^0) \). Also, one can easily show that: \( SP^0 > SP^0 \) if and only if \( \delta > 0.5 \). In other words, if the market maker is pessimistic \( (\delta > 0.5) \), s/he sets a lower spread when the noise trader is optimistic \( (SP^0) \), than the spread s/he sets when the noise trader is also pessimistic \( (SP^0) \). Q.E.D.

**Proof of Proposition 5.** We prove the case of \( \psi = 0 \), the other cases are parallel. Let:
\[ P_N = \gamma (1 - e^5) + (1 - \gamma) (1 - e^8), \] \[ P_S = ye^5, \] and \( P_b = (1 - \gamma) e^8. \)

We have:
\[ I_{SP}(Q) = \sum_{Q} P^0(Q) \log \frac{P^0(Q)}{Q_{P}^0} = P^0(N) \log \frac{1}{1 - \mu} + P^0(B) \log \frac{P_B}{P_B}, \]
\[ + P^0(S) \log \frac{P_S}{P_S}. \]
and

\[
I_{\rho}\left(p^t\right) = \sum_{Q} p^Q(Q) \log \frac{p^Q(Q)}{p^Q(\hat{Q})} - p^Q(\hat{Q}) \log \frac{1}{1-\mu} + p^Q(B) \log \frac{p^Q(B)}{p^Q(\hat{B})} + p^Q(S) \log \frac{p^Q(S)}{p^Q(\hat{S})}.
\]

Following Eq. (5), we apply Bayes Law to find posterior probabilities, for every \(t\):

\[
\eta_{t+1} = \left(1-\alpha\right)\eta^t + \left(1-\alpha\right)\left(1-\gamma\right)\eta^t \left(1-\gamma\right)\theta^t + \left(1-\alpha\right)\left(1-\gamma\right)\left(1-\gamma\right)\theta^t \left(1-\gamma\right)\eta^t.
\]

Similarly, we obtain \(\theta_{t+1}\) and \(\theta_{t+1}\), and compute

\[
\eta_{t+1} = \alpha\eta^t + \left(1-\alpha\right)\left(1-\gamma\right)\eta^t \left(1-\gamma\right)\theta^t + \left(1-\alpha\right)\left(1-\gamma\right)\left(1-\gamma\right)\theta^t \left(1-\gamma\right)\eta^t.
\]

This yields the following equation

\[
\log \frac{\eta_{t+1}}{\eta_{t+1}} = \log \frac{\eta_0}{\eta_0} + \left[ n_t \log \left(\frac{P^t(\hat{Q})}{P^t(\hat{Q})} + S_t \log \left(\frac{P^t(S)}{P^t(B)}\right) \right) - n_t \log P^t_\mu + S_t \log P^t_\mu + B_t \log P^t_\mu \right].
\]

By the Strong Law of Large Numbers

\[
\frac{1}{t} \log \frac{\eta_{t+1}}{\eta_{t+1}} = \alpha \sum_{Q} p^Q(Q) \log P^Q(Q) - \sum_{Q} p^Q(Q) \log P^Q(Q) = -I_{\rho}\left(p^t\right).
\]

Similarly

\[
\frac{1}{t} \log \frac{\theta_{t+1}}{\theta_{t+1}} = \alpha \sum_{Q} p^Q(Q) \log P^Q(Q) - \sum_{Q} p^Q(Q) \log P^Q(Q) = -I_{\rho}\left(p^t\right).
\]

Since \(I_{\rho}\left(p^t\right)\) is always nonnegative, and it takes a value of zero if and only if \(p^t = p^t\), the ratio \(\frac{\eta_{t+1}}{\eta_{t+1}}\) converges almost surely to 0 at an exponential rate \(-I_{\rho}\left(p^t\right)\). Similarly, \(\frac{\theta_{t+1}}{\theta_{t+1}}\) converges almost surely to 0 at an exponential rate \(-I_{\rho}\left(p^t\right)\).

Given the equilibrium quote from Eq. (6), we have

\[
b_t - V_S = \theta_t \alpha V + \theta_t \alpha V_S - V_S = \left(\alpha V - \alpha V_S\right) / \eta_t = \theta_t \Delta - \theta_t (1-\delta).
\]

Since \(\theta_t\) converges almost surely to 0 at an exponential rate \(-I_{\rho}\left(p^t\right)\) and \(\theta_t\) converges almost surely to 0 at an exponential rate \(-I_{\rho}\left(p^t\right)\), \(b_t\) converges almost surely to \(V^t\) at an exponential rate \(-\min[I_{\rho}\left(p^t\right), I_{\rho}\left(p^t\right)]\).

Proof of Proposition 6. Recall that \(\sigma_\rho\) does not directly appear in the expression for the rates of convergence. Thus, we need to look at its impact through \(\phi\) and \(\phi\). We assume that uninformed traders are equally likely to buy or sell: \(\gamma^2 = (1-\gamma)^2\). Under this assumption we show that \(I_{\rho}\left(p^t\right)\) increases in \(\eta\) and \(\phi\), and \(I_{\rho}\left(p^t\right)\) increases in \(\eta\) and \(\phi\):

\[
\frac{\partial I_{\rho}\left(p^t\right)}{\partial \eta} = \mu(1-\eta) \frac{p^\mu p^\mu(S) - p^\mu p^\mu(B)}{p^\mu(S)p^\mu(B)} > 0,
\]

and

\[
\frac{\partial I_{\rho}\left(p^t\right)}{\partial \phi} = \mu(1-\phi) \frac{-p^\mu p^\mu(S) + p^\mu p^\mu(B)}{p^\mu(S)p^\mu(B)} > 0.
\]

Thus, when uninformed traders are equally likely to buy or sell, the entropy increases in \(\eta\):

\[
\frac{\partial I_{\rho}\left(p^t\right)}{\partial \eta} = \mu(1-\eta) \frac{p^\mu p^\mu(S) + p^\mu p^\mu(B)}{p^\mu(S)p^\mu(B)} > 0.
\]

and

\[
\frac{\partial I_{\rho}\left(p^t\right)}{\partial \phi} = \mu(1-\phi) \frac{-p^\mu p^\mu(S) + p^\mu p^\mu(B)}{p^\mu(S)p^\mu(B)} > 0.
\]

Since both \(\phi\) and \(\phi\) are cumulative distribution functions, they both decrease in \(\sigma_\rho\). Hence, the above results imply that the higher the uncertainty regarding noise traders’ sentiment the lower the speed of price reversion. Q.E.D.

Proof of Proposition 7. The rate of convergence is give by

\[
I_{\rho}\left(p^t\right) = \left(\eta + (1-\eta)\psi\right) \log \left[\frac{\eta + (1-\eta)\psi}{(1-\eta)(1-\psi)}\right] + \left[\eta + (1-\eta)(1-\phi)\right] \log \left[\eta + (1-\eta)(1-\phi)\right]\right].
\]

When the noise trader is highly optimistic \((\rho \to \infty)\), we have: \(\rho \to \infty, \psi \to 0,\) and \(\phi \to 1\). This implies:

\[
I_{\rho}\left(p^t\right) \to \log 1/\eta.
\]

When the noise trader is highly pessimistic \((\rho \to -\infty)\), we have: \(\psi \to 1,\) and \(\phi \to 0\). This implies:

\[
I_{\rho}\left(p^t\right) \to -\infty.
\]

Therefore, given that a high signal has materialized, the rate of convergence will be higher if noise traders are overly pessimistic than when they are overly optimistic: \(I_{\rho}\left(p^t\right) > I_{\rho}\left(p^t\right)\). Similarly, given that a low signal has materialized, the rate of convergence will be higher if noise traders are overly optimistic than when they are overly pessimistically \(I_{\rho}\left(p^t\right) I_{\rho}\left(p^t\right)\). Q.E.D.

References


Wang, F. Albert (2002). Trading on noise as if it were information: Price, liquidity, volume and profit. Working paper, Rice University.